SOME ASPECTS OF NON-LINEAR OPTIMIZATION

A PROJECT REPORT

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Certificate

This is to certify that the project report entitled **SOME ASPECTS OF NON-LINEAR OPTIMIZATION** submitted by **SASMITA PATEL** to the National Institute of Technology Rourkela, Orissa for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

May, 2014

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ABSTRACT

We provide a concise introduction to some methods for solving nonlinear optimization problems. This dissertation includes a literature study of the formal theory necessary for understanding optimization and an investigation of the algorithms available for solving a special class of the non-linear programming problem, namely the quadratic programming problem. It was not the intention of this dissertation to discuss all possible algorithms for solving the quadratic programming problem, therefore certain algorithms for convex and non-convex quadratic programming problems . Some of the algorithms were selected arbitrarily, because limited information was available comparing the efficiency of the various algorithms.

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CHAPTER 1

1 Introduction

The term 'non linear programming' usually refers to the problem in which the objective function becomes non-linear, or one or more of the constraint inequalities have non-linear or both.The solution of nonlinear optimization problems that is the minimization or maximization of an objective function involving unknown parameters/variables in which the variables may be restricted by constraints is one of the core components of computational mathematics. Nature (and man) loves to optimize, and the world is far from linear. In Applied Mathematics, the eminent mathematician Gil Strange opines that optimization, along with the solution of systems of linear equations, and of (ordinary and partial) differential equations, is one of the three cornerstones of modern applied mathematics.

CHAPTER 2

2 Preliminaries

2.1 Local Optimal Point

A point x^* is said to be a local optimal point, if there exist no point in the neighbourhood of x^* which is better than x^*. similarly a point x^* is a local minimal point if no point in the neighborhood has a function value smaller than $f(x^*)$.

2.2 Global Optimal Point

A point x^{**} is said to be a Global optimal point , if there is no point in the entire search space which is better than the point x^{**}. similarly a point x^{**} is a Global minimal point if no point in the entire search space has a function value smaller than $f(x^{**})$.

2.3 Convex Function

A function $f(x)$ is said to be convex function over the region $'S'$. If for any two points $x_1, x_2 \in S$ we have the function

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

where $0 \leq \lambda \leq 1$
Strictly convex function means

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2)$$

where $0 \leq \lambda \leq 1$

CHAPTER 3

3 NON-LINEAR OPTIMIZATION

3.1 Introduction

The Linear Programming Problem which can be review as to

$$maximize \;\; Z = \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$subject\ to \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad for \quad i = 1, 2, ..., m \tag{2}$$

$$and \;\; x_j \geq 0 \qquad for \quad j = 1, 2, ..., n \tag{3}$$

The term 'non linear programming' usually refers to the problem in which the objective function (1) becomes non-linear, or one or more of the constraint inequalities (2) have non-linear or both.

3.2 General Non-Linear Programming Problem

The mathematical formulation of general non linear programming problem may be expressed as follows:

max.(or min.)z=$C(x_1, x_2, ..., x_n)$,subject to the constraints:

$$\begin{gathered} a_1(x_1, x_2, ..., x_n)\{\leq, = or \geq\}b_1 \\ a_2(x_1, x_2, ..., x_n)\{\leq, = or \geq\}b_2 \\ a_3(x_1, x_2, ..., x_n)\{\leq, = or \geq\}b_3 \\ .... \\ a_m(x_1, x_2, ..., x_n)\{\leq, = or \geq\}b_m \\ \text{and } x_j \geq 0, j = 1, 2, ..., n \end{gathered}$$

where either $C(x_1, x_2, ..., x_n)$ or some $a_i(x_1, x_2, ..., x_n), i = 1, ..., m$; or both are non-linear. In matrix notation,the general non-linear programming problem may be written as follows:

$$\begin{gathered} \text{max.(or min.)} z = C(X), \text{ subject to the constraints:} \\ a_i(X)\{\leq, = or, \geq\}b_i, i = 1, 2, ..., m \\ \text{and } X \geq 0, \end{gathered}$$

where either $C(X)$ or some $a_i(X)$ or both are non linear in X.

For Example:
$max.Z = 2x_1 + 3x_2$
subject to ${x_1}^2 + {x_2}^2 \leq 20$, and $x_1, x_2 \geq 0$

3.3 Unconstrained Optimization

We use to.

$$optimize \ \ f(x_1, x_2, ..., x_n)$$

The extreme points from the solution

$\dfrac{\partial f}{\partial x_1} = 0$

$\dfrac{\partial f}{\partial x_2} = 0$

$\vdots$

$\dfrac{\partial f}{\partial x_n} = 0$

For one Variable

$\dfrac{\partial^2 f}{\partial x^2} > 0$ Then f is minimum.

$\dfrac{\partial^2 f}{\partial x^2} < 0$ Then f is maximum.

$\dfrac{\partial^2 f}{\partial x^2} = 0$ Then further investigation needed.

For two variable

$rt - s^2 > 0$ Then the function is minimum.
$rt - s^2 < 0$ Then the function is maximum.
$rt - s^2 = 0$ Further investigation needed.

Where $r = \dfrac{\partial^2 f}{\partial {x_1}^2}, s = \dfrac{\partial^2 f}{\partial x_1 x_2}, t = \dfrac{\partial^2 f}{\partial {x_2}^2}$

For 'n' Variable

Hessian Matrix

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial {x_1}^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots\cdots\cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial {x_2}^2} & \cdots\cdots\cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n} & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots\cdots\cdots & \dfrac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$$

$det \ \ (H) > 0$ at p_1,f is attains minimum at p_1.
$det \ \ (H) < 0$ at p_1,f is attains maximum at p_1.

3.4 Constrained Optimization

$optimize f(x_1, x_2, ..., x_n)$
subject to $g_i(x_1, x_2, ..., x_n) = 0, i = 1(1)m$
and $x_i \geq 0 \quad \forall i$
For solving this we use Lagrange Multiplier Method and Kuhn-Tucker Condition.

3.4.1 Lagrange Multiplier Method

Here we discussed the optimization problem of continuous functions. The non-linear programming problem is composed of some differentiable objective function and equality side constraints, the optimization may be achieved by the use of Lagrange multipliers.
A Lagrange multiplier measures the sensitivity of the optimal value of the objective function to change in the given constraints b_i in the problem.

Consider the problem of determining the global optimum of

$$Z = f(x_1, x_2, ..., x_n)$$

subject to the constraints

$$g_i\ (x_1, x_2, ..., x_n)\ =\ b_i\ ,\ i = 1, 2, ..., m.$$

Let us first formulate the Lagrange function L defined by:

$$L(x_1, ..., x_n; \lambda_1, ..., \lambda_m)\ =\ f(x_1, ..., x_n) + \lambda_1 g_1(x_1, ..., x_n) + \lambda_2 g_2(x_1, ..., x_n) + ... + \lambda_m g_m(x_1, ..., x_n)$$

where$\lambda_1, \lambda_2, ..., \lambda_m$ are called Lagrange Multiplier.
For the stationary points

$$\frac{\partial L}{\partial x_j} = 0 \quad , \quad \frac{\partial L}{\partial \lambda_i} = 0 \quad \forall\ j = 1(1)n \quad \forall\ i = 1(1)m$$

Solving the above equation to get stationary points.

Example:1

$min. f(x_1, x_2) = {x_1}^2 + {x_2}^2$
$x_1 x_2 = 1$
$and\ x_1, x_2 \geq 0$
solution: form lagrangian multiplier function

$$L(x_1, x_2, \lambda) = {x_1}^2 + {x_2}^2 + \lambda(x_1 x_2 - 1)$$

for the stationary points

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + \lambda x_2 = 0 \quad (4)$$
$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + \lambda x_1 = 0 \quad (5)$$
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 x_2 - 1 = 0 \quad (6)$$

from equation (4) and (5)

$x_1 = x_2 \;\; and \;\; x_1 = -x_2$

substituting $x_1 = x_2$ in (6) we get

$x_1 x_2 - 1 = 0$
${x_1}^2 = 1$
$x_1 = 1 \;\; and \;\; x_2 = -1$ (not possible)
$x_1 = 1$
$min.f = 2$

3.4.2 Kuhn-Tucker Condition

Consider The Convex Non-linear programming problem

$$min.f(X)$$

subject to

$$g_i(X) \leq 0, \qquad i = 1(1)m$$

and

$$X \geq 0$$

Form $L(X, \overline{\lambda}) = f(X) + \overline{\lambda}^T G(X)$

where $\overline{\lambda}^T = [\lambda_1, \lambda_2, ..., \lambda_n]$

$G(X) = [g_1(X), g_2(X), ..., g_n(X)]^T$

The necessary and sufficient condition for X^* to be the optimal solution of above CNLPP are

$$\begin{aligned} \frac{\partial L}{\partial X_j}_{(X^*, \overline{\lambda}^*)} &\geq 0 \quad j = 1(1)n \\ X_j \frac{\partial L}{\partial X_j}_{(X^*, \overline{\lambda}^*)} &= 0 \quad j = 1(1)n \\ \frac{\partial L}{\partial \lambda_i}_{(X^*, \overline{\lambda}^*)} &\leq 0 \quad i = 1(1)m \\ \lambda_i \frac{\partial L}{\partial \lambda_i}_{(X^*, \overline{\lambda}^*)} &= 0 \quad i = 1(1)m \end{aligned}$$

Example.1

Solve the following NLPP using the Kuhn-Tucker conditions:

$maximize\ \ Z = 2{x_1}^2 - 7{x_2}^2 + 12x_1x_2,$
subject to

$$2x_1 + 5x_2 \leq 98$$

$$x_1, x_2 \geq 0$$

Solution:

Let
$f(X) = 2{x_1}^2 - 7{x_2}^2 + 12x_1x_2$
$h(X) = 2x_1 + 5x_2 - 98$

The Kuhn-Tucker condition are

$\dfrac{\partial f}{\partial x_1} - \lambda \dfrac{\partial h}{\partial x_1} = 0$
$\dfrac{\partial f}{\partial x_2} - \lambda \dfrac{\partial h}{\partial x_2} = 0$
$\lambda h(X) = 0,$
$h(X) \leq 0,$
$\lambda \geq 0.$

Applying these conditions, we get

$$4x_1 + 12x_2 - 2\lambda = 0 \tag{7}$$
$$12x_1 - 14x_2 - 5\lambda = 0 \tag{8}$$
$$\lambda(2x_1 + 5x_2 - 98) = 0 \tag{9}$$
$$2x_1 + 5x_2 - 98 \leq 0 \tag{10}$$
$$\lambda \geq 0 \tag{11}$$

From equation (9) either $\lambda = 0$ or $2x_1 + 5x_2 - 98 = 0$.
When $\lambda = 0$, equations (7) and (8) give $x_1 = x_2 = 0$, Which does not satisfy condition (10).
Thus a feasible solution cannot be obtained for $\lambda = 0$.
When $2x_1 + 5x_2 - 98 = 0$, this equation along with (7) and (8) gives the solution, $x_1 = 44$ and $x_2 = 2$ with $\lambda = 100$ and $Z_{max} = 4900$

CHAPTER 4

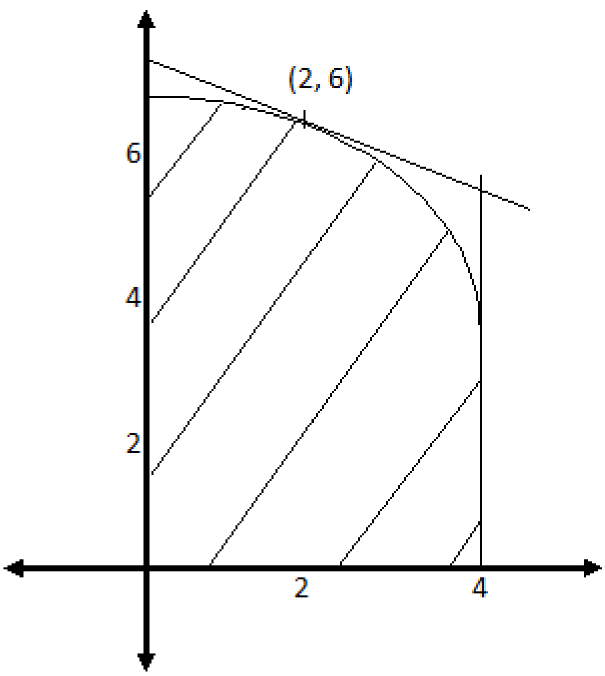
4 GRAPHICAL SOLUTION OF NLPP

In a linear programming problem, the optimal solution was usually obtained at one of the extreme points of the convex region generated by the constraints and the objective function of the problem. But, it is not necessary to find the solution at a corner or edge of the feasible region of non-linear programming problem.

example-1 :
solve graphically the following problem
maximize$z = 3x_1 + 5x_2$
subject to
$x_1 \leq 4$
$9{x_1}^2 + 5{x_2}^2 \leq 216$
and $x_1, x_2 \geq 0$
solution:
$x_1 = 4$

$$9{x_1}^2+5{x_2}^2 = 216 \tag{1}$$

$$\Rightarrow \frac{{x_1}^2}{\frac{216}{9}} + \frac{{x_2}^2}{\frac{216}{5}} = 1$$



it is a equation of ellipse let $3x_1 + 5x_2 = k$
now differentiate the equation with respect to x_1

$$3 + 5\frac{dx_2}{dx_1} = 0$$

$$\Rightarrow \frac{dx_2}{dx_1} = -\frac{3}{5} \tag{2}$$

Again differentiate equation(1)with respect to x_1

$9{x_1}^2 + 5{x_2}^2 = 216$

$\Rightarrow 18x_1 + 10x_2 \frac{dx_2}{dx_1} = 0$

$$\Rightarrow \frac{dx_2}{dx_1} = -\frac{18x_1}{10x_2} \tag{3}$$

put (3) in (2) we get

$-\frac{18x_1}{10x_2} = -\frac{3}{5}$

$\Rightarrow 3x_1 = x_2$ put the value of x_2 in equation(1) we get

$x_1 = -2$ or $x_1 = 2$

when$x_1 = 2$, $x_2 = 6$

$max z = 36$

CHAPTER 5

5 QUADRATIC PROGRAMMING

5.1 Introduction

Quadratic programming problems are very similar to linear programming problems. The main difference between the two is that our objective function is now a quadratic function (the constraints are still linear).Because of its many applications, quadratic programming is often viewed as a discipline in and of itself. More importantly, though, it forms the basis of several general nonlinear programming algorithms. We begin this section by examining the Karush-Kuhn-Tucker conditions for the QP and see that they turn out to be a set of linear equalities and complementarity constraints. Much like in separable programming, a modified version of the simplex algorithm can be used to find solutions.

Quadratic Form

A Homogeneous polynomial of degree two in any number of variables is called a Quadratic forms or an expression of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

is called Quadratic form of n variable.

5.2 General Quadratic Programming

Definition. let x^T and $c \in R^n$ and, Q be a symmetric $n \times n$ real matrix. Then,

$$max\ f(X) = CX + \frac{1}{2}X^T QX$$

subject to

$$AX \leq b$$

and

$$X \geq 0$$

.
where $b^T \in R^m$ and A be $m \times n$ real matrix is called a general quadratic programming problem.

The function $X^T QX$ defines a quadratic form with Q being a symmetric matrix. The quadratic form $X^T QX$ is said to be positive definite if $X^T QX > 0$ for $X \neq 0$ and positive-semi-definite if $X^T QX \geq 0$ for all X such that there is one $X \neq 0$ satisfying $X^T QX = 0$. Similarly, $X^T QX$ is said to be negative definite and negative-semi-definite if $-X^T QX$ is positive-definite and positive-semi-definite respectively. The function $X^T QX$ is assumed to be negative-definite in the maximization case, and positive definite in the minimization case.

5.3 Wolfe's Method

Let the quadratic programming problem be :

$$Maximize \quad z = f(X) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$$

subject to the constraints :

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, x_j \geqslant 0 (i = 1, ..., m, j = 1, ..., n)$$

Where $c_{jk} = c_{kj}$ for all j and k,$b_i \geqslant 0$ for all i = 1,2,...,m.
Also, assume that the quadratic form

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$$

be negative semi-definite.
Then,the Wolfe's iterative procedure may be outlined in the following steps:

Step 1. First,convert the inequality constraints into equations by introducing slack-variables q_i^2 in the ith constraint (i = 1,...,m) and the slack variables r_j^2 in the jth non-negatively constraint (j = 1,2,...,n).

Step 2. Then, construct the Lagrangian function

$$L(X, q, r, \lambda, \mu) = f(X) - \sum_{i=1}^{m} \lambda_i [\sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2] - \sum_{j=1}^{n} \mu_j [-x_j + r_j^2]$$

Where $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{q} = (q_1^2, q_2^2, ..., q_m^2)$, $\mathbf{r} = (r_1^2, r_2^2, ..., r_n^2)$, and
$\lambda = (\lambda_1, \lambda_2, ...\lambda_m)$ $\mu = (\mu_1, \mu_2, \ ... \ \mu_n)$

Differentiating the above function '**L**' partially with respect to the components of x, q, r, λ, μ and equating the first order partial derivatives to zero, we derive Kuhn-Tucker conditions from the resulting equations.

Step 3. Now introduce the non-negative artificial variable $v_j, j = 1, 2, ..., n$ in the Kuhn-Tucker conditions

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0$$

for j=1,2,...,n and to construct an objective function

$$z_v = v_1 + v_2 + ... + v_n$$

Step 4. In this step, obtain the initial basic feasible solution to the following linear programming problem :

$$\textbf{Min. } z_v = v_1 + v_2 + ... + v_n.$$

Subject to the constraints :

$$\sum_{k=1}^{n} c_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + v_j = -c_j (j = 1, ...n)$$

$$\sum_{j=1}^{n} a_{ij}x_j + q_i^2 = b_i (i = 1, ..., m)$$

$$v_j, \lambda_j, \mu_j, x_j \geqslant 0 \; (i = 1, ..., m; j = 1, ..., n)$$

and satisfying the complementary slackness condition:

$$\sum_{j=1}^{n} \mu_j x_j + \sum_{i=1}^{m} \lambda_i s_i = 0, \; (where \; s_i = q_i^2)$$

or

$$\lambda_i s_i = 0 \quad \mu_j x_j = 0 \quad (for \; i = 1, ..., m; j = 1, ..., n).$$

Step 5. Now, apply two-phase simplex method in the usual manner to find an optimum solution to the LP problem constructed in Step 4. The solution must satisfy the above complementary slackness condition.

Step 6. The optimum solution thus obtained in Step 5 gives the optimum solution of given QPP also.

Important Remarks:

1.If the Quadratic Programming Problem is given in the minimization form, then convert it into maximization one by suitable modifications in $f(X)$ and the '$\geqslant$' constraints.

2. The solution of the above system is obtained by using *Phase I* of simplex method. The solution does not require the consideration of *Phase II*. Only maintain the condition $\lambda_i s_i = 0 = \mu_j x_j$ all the time.

3. It should be observed that *Phase I* will end in the usual manner with the sum of all artificial variables *equal to zero* only if the feasible solution to the problem exists.

Example.1

$min. 2{x_1}^2 + 2{x_2}^2 - 4x_1 - 4x_2$
subject to $2x_1 + 3x_2 \leq 6$
and $x_1, x_2 \geq 0$

solution

Since the given objective function is convex and each constraint is convex therefore the given NLPP is a CNLPP.

Now $L(X, \overline{\lambda}) = 2{x_1}^2 + 2{x_2}^2 - 4x_1 - 4x_2 + \lambda(2x_1 + 3x_2 - 6)$

Therefore the kuhn-tucker condition are

$$\frac{\partial L}{\partial x_j} \geq 0 \Rightarrow 4x_1 - 4 + 2\lambda \geq 0$$

$$\Rightarrow 4x_1 - 4 + 2\lambda - \mu_1 = 0 \qquad (1)$$

$$4x_2 - 4 + 3\lambda \geq 0$$

$$\Rightarrow 4x_2 - 4 + 3\lambda - \mu_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} \leq 0 \Rightarrow 2x_1 + 3x_2 - 6 \leq 0$$

$$2x_1 + 3x_2 - 6 + S_1 = 0 \qquad (3)$$

$$x_j \frac{\partial L}{\partial x_j} = 0 \Rightarrow x_1\mu_1 = 0, x_2\mu_2 = 0 \qquad (4)$$

$$\lambda_i \frac{\partial L}{\partial \lambda} = 0 \qquad (5)$$

The above system of equation can be written as
$4x_1 + 2\lambda - \mu_1 = 4$
$4x_2 + 3\lambda - \mu_2 = 4$
$2x_1 + 3x_2 + S_1 = 6 \qquad (6)$
So we have to find $x_1, x_2, \lambda_1, \mu_1, \mu_2$ and S_1
such that $x_1\mu_1 = 0, x_2\mu_2 = 0, \lambda S_1 = 0$ where $x_1, x_2, \lambda, \mu_1, \mu_2, S_1 \geq 0$
This equation (6) is a LPP with out an objective function. To find the solution we can write (6) as the following LPP.
$max. Z = -R_1 - R_2$
subject to

$$4x_1 + 2\lambda - \mu_1 + R_1 = 4$$

$$4x_2 + 3\lambda - \mu_2 + R_2 = 4$$

$$2x_1 + 3x_2 + S_1 = 6$$

$$x_1, x_2, \lambda, \mu_1, \mu_2, S_1 \geq 0$$

Now solve this by the two phase simplex method. The end of the phase (1) gives the feasible solution of the problem
The optimal solution of the QPP
$x_1 = 1, x_2 = 1, S_1 = 1, \mu_1 = \mu_2 = \lambda = 0$

5.4 Beale's Method

In beale's method we solve Quadratic Programming problem and in this method we does not use the Kuhn-Tucker condition. At each iteration the objective function is expressed in terms of non basic variables only.

Let the QPP be given in the form

$$max.f(X) = CX + \frac{1}{2}X^TQX$$

subject to the constraints $AX = b, X \geq 0$.
Where

$$\begin{aligned} X &= (x_1, x_2, ..., x_{n+m}) \\ c \quad &is \quad 1 \times n \\ A \quad &is \quad m \times (n+m) \\ and \quad Q \quad &is \quad symmetric. \end{aligned}$$

With out loss of generality, every QPP with linear constraints can be written in this form.

Algorithm

Step 1
First express the given QPP with Linear constraints in the above form by introducing slack and surplus variables,etc.

Step 2
Now select arbitrarily m variables as basic and remaining n variables as non-basic.
Now the constraints equation $AX = b$ can be written as

$$BX_B + RX_{NB} = b \Rightarrow X_B = B^{-1}b - B^{-1}RX_{NB}$$

Where
X_B-Basic vector
X_{NB}-Non-basic vector
and the matrix A is partitioned to sub matrices B and R corresponding to X_B and X_{NB} respectively.

Step 3

Express the objective function f(x) in terms of X_{NB} only using the given and additional constraint , if any . Thus we observe that by increasing the value of any of the non-basic variables, the value of the objective function can be improved. Now the constraints on the new problem become

$$B^{-1}RX_{NB} \leq B^{-1}b \quad (since \ \ X_B \geq 0)$$

Thus, any component of X_{NB} can increase only until $\dfrac{\partial f}{\partial x_{NB}}$ becomes zero or one or more components ofX_B are reduced to zero.

Step 5
Now we have $m+1$ non-zero variables and $m+1$ constraints which is a basic solution to the extended set of constraints.

Step 6
We go on repeating the above procedure until no further improvement in the objective function may be obtain by increasing one of the non-basic variable.

Example:

Use Beale's Method to solve following problem
$max.Z = 4x_1 + 6x_2 - 2{x_1}^2 - 2x_1x_2 - 2{x_2}^2$
suject to $x_1 + 2x_2 \leq 2$

$$and \ \ x_1, x_2 \geq 0$$

Solution:
Step:1

$$Max.Z = 4x_1 + 6x_2 - 2{x_1}^2 - 2x_1x_2 - 2{x_2}^2 \tag{1}$$

subject to

$$x_1 + 2x_2 + x_3 = 2 \tag{2}$$

and $x_1, x_2, x_3 \geq 0$
taking $X_B = (x_1); X_{NB} = \binom{x_2}{x_3}$
and $x_1 = 2 - 2x_2 - x_3$ (3)

Step:2
put (3) in (1), we get
$max.f(x_2, x_3) = 4(2 - 2x_2 - x_3) + 6x_2 - 2(2 - 2x_2 - x_3)^2 - 2(2 - 2x_2 - x_3)x_2 - 2{x_2}^2$
$\dfrac{\partial f}{\partial x_2} = -2 + 8(2 - 2x_2 - x_3) + 8x_2 - 4x_2 - 2(2 - x_3)$
$\dfrac{\partial f}{\partial x_3} = -4 + 4(2 - 2x_2 - x_3) + 2x_2$
Now $\dfrac{\partial f}{\partial x_2}_{(0,0)} = 10$
$\dfrac{\partial f}{\partial x_3}_{(0,0)} = 4$

Here '+ve' value of $\dfrac{\partial f}{\partial x_i}$ indicates that the objective function will increase if x_i increased . Similarly '-ve' value of $\dfrac{\partial f}{\partial x_i}$ indicates that the objective function will decrease if x_i is decrease .
Thus, increase inx_2 will give better improvement in the objective function.

Step:3
$f(x)$ will increase if x_2 increased .
If x_2 is increased to a value greater then 1, x_1 will be negative.
Since $x_1 = 2 - 2x_2 - x_3$
$x_3 = 0; \dfrac{\partial f}{\partial x_2} = 0$

$$\Rightarrow 10 - 12x_2 = 0$$

$$\Rightarrow x_2 = \frac{5}{6}$$

$min.(1, \dfrac{5}{6}) = \dfrac{5}{6}$

The new basic variable is x_2.

Second Iteration:

Step:1
let $X_B = (x_2)$ $\quad\quad X_{NB} = \binom{x_1}{x_3}$

$$x_2 = 1 - \frac{1}{2}(x_1 + x_3)$$

Step:2
Substitute (4) in (1)
$max.f(x_1, x_3) = 4x_1 + 6(1 - \dfrac{1}{2}(x_1 + x_3)) - 2{x_1}^2 - 2x_1(1 - \dfrac{1}{2}(x_1 + x_3)) - 2(1 - \dfrac{1}{2}(x_1 + x_3))^2$
$\dfrac{\partial f}{\partial x_1} = 1 - 3x_1, \quad\quad \dfrac{\partial f}{\partial x_3} = -1 - x_3$
$\dfrac{\partial f}{\partial x_1}_{(0,0)} = 1$
$\dfrac{\partial f}{\partial x_3}_{(0,0)} = -1$
This indicates that x_1 can be introduce to increased objective function.

Step:3
$x_2 = 1 - \dfrac{1}{2}(x_1 + x_3)\ \ and\ \ x_3 = 0$
If x_1 is increased to a value greater then 2 ,x_2 will become negative.
$\dfrac{\partial f}{\partial x_1} = 0$
$\Rightarrow 1 - 3x_1 = 0$
$\Rightarrow x_1 = \dfrac{1}{3}$

$min.(2, \frac{1}{3}) = \frac{1}{3}$

Therefore $x_1 = \frac{1}{3}$

Hence $x_1 = \frac{1}{3}, \qquad x_2 = \frac{5}{6}, \qquad x_3 = 0$

and $max.f(x) = \frac{25}{6}$

CHAPTER 6

6 SEPARABLE PROGRAMMING

6.1 Introduction

Separable Programming deals with such non linear programming problems in which the objective function as well as all the constraints are separable.We solve an optimization problem with non-linear objective function and linear constraints by a linear program. Later we will extend the idea to solve, at least approximately, convex optimization problems with separable non-linear objective functions and convex feasible sets defined by separable convex functions. The technique, called separable programming, basically replaces all separable convex functions, in objectives and constraints, by piecewise linear convex functions.

6.2 Separable Function

Definition:A Function $f(x_1, x_2, ..., x_n)$ is said to be separable if it can be expressed as the sum of n single valued functions $f_1(x_1), f_2(x_2), ..., f_n(x_n)$, i.e.

$$f(x_1, x_2, ..., x_n) = f_1(x_1) + f_2(x_2) + ... + f_n(x_n)$$

.

Example:

$g(x_1, x_2, ..., x_n) = c_1x_1 + ... + c_nx_n$ where $c\,'s$ are constants,
is a separable function.

$g(x_1, x_2, x_3) = {x_1}^3 + {x_2}^2 log(x_1 + x_3) + x_3 3^{x_2} + e^{(x_1x_3)}$
is not a separable function.

Reducible to Separable Form: Sometimes the functions are not directly separable but can be made separable by simple substitutions.

Example:
$max.Z = x_1{x_2}^2$
Let $y = x_1{x_2}^2$
Then $logy = logx_1 + 2logx_2$
Hence the problem becomes

$$max.Z = Y$$

subject to

$$logy = logx_1 + 2logx_2$$

which is separable.

6.3 Separable Programming Problem

A NLPP in which the objective function can be expressed as a linear combination of several different single variable functions, of which some or all are non-linear, is called a separable programming problem.

6.4 Reduction of separable programming problem to linear programming problem

Let us consider the separable programming problem

$$Max.(Min.)z = \sum_{j=1}^{n} f_j(x_j)$$

subject to the constraints :

$$\sum_{j=1}^{n} g_{ij}(x_j) \leqslant b_i$$
$$x_j \geqslant 0 \quad (i = 1, 2, ..., m; j = 1, 2, ..., n)$$

where some or all $g_{ij}(x_j), f_j(x_j)$ are non linear.
Then the equivalent mixed problem is

$$Max.(or\ Min.)z = \sum_{j=1}^{n} \sum_{k=1}^{n} f_j(a_{jk})w_{jk}$$

subject to the constraints:

$$\sum_{j=1}^{n} \sum_{k=1}^{n} g_{ij}(a_{jk})w_{jk} \leqslant b_i \quad , i = 1, 2, ..., m$$

$$\begin{aligned}
&0 \leqslant w_{j1} \leqslant y_{j1} \\
&0 \leqslant w_{jk} \leqslant y_{j.k-1} + y_{jk} \quad where \;\; k = 2, 3, ..., k_{j-1} \\
&0 \leqslant w_{jk_j} \leqslant y_{j.k_{j-1}} \\
&\sum_{k=1}^{k_j} w_{jk} = 1, \sum_{k=1}^{k_{j-1}} y_{jk} = 1 \\
&y_{jk} = 0 \quad or \quad 1 \;\; ; \qquad k = 1, 2, ..., k_j \;, \qquad j = 1, 2, ..., n
\end{aligned}$$

The variables for the approximating problem are given by w_{jk} and y_{jk} .

We can use the regular simplex method for solving the approximate problem under the additional constraints involving y_{jk} .

6.5 Separable Programming Algorithm

Step-1
If the objective function is of minimization form, convert it into maximization.

Step-2
Test whether the functions $f_j(x_j)$ and $g_{ij}(x_j)$ satisfy the concavity (convexity) conditions required for the maximization(minimization) of non-linear programming problem. If the condition are not satisfied, the method is not applicable, otherwise go to next step.

Step-3
Divide the interval $0 \leqslant x_j \leqslant t_j \quad (j = 1, 2, ..., n)$ into a number of mesh points $a_{jk} \ (k = 1, 2, ..., K_j)$ such that $a_{ij} = 0, a_{j1} < a_{j2} < ... < a_{jK_j} = t_j$

Step-4
For each point a_{jk} , compute piecewise linear approximation for each $f_j(x_j)$ and $g_{ij}(x_j)$ where $j = 1, 2, ..., n; i = 1, 2, ..., m$.

Step-5
Using the computations of step-4,write down the piece-wise linear approximation of the given NLPP.

Step-6
Now solve the resulting LPP by two-phase simplex method. For this method consider $w_{i1} \quad (i = 1, 2, ..., m)$ as artificial variables. Since, the costs associated with them are not given, we assume them to be zero.
Then,Phase-I of this method is automatically complete.
Therefore, the initial simplex table of Phase-I is optimum and hence will be the starting simplex table for Phase-II.

Step-7
Finally, we obtain the optimum solution x_j^* of the original problem by using the relations:

$$x_j^* = \sum_{k=1}^{K_j} a_{jk} w_{jk} \qquad (j = 1, 2, ..., n).$$

References

[1] S.D.Sharma, Operations Research, KNRN publication.

[2] Prem Kumar Gupta and Dr.D.S. Hira, Operation Research, S.Chand Publication.

[3] F.S.Hillier and Gerald J. Lieberman , Operations Research.

[4] Martina Vankova, Algorithms for the Solution of the Quadratic Programming Problem, January 2004